Some adaptive economic processes in social interactions*

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Abstract

On the human brain activities in social interaction, there may be associated with some kind of reinforcement mechanism like crowd psychology or mimicking drive, or sometimes, creating a social mechanism like punishments. Some of such reinforcement mechanisms can be interpreted with a nonlinear Polya urn process of which W. Brian Arthur(1994) is fond. He called structure as emergence of a new combination of proportion generated by the stochastic process. In the case of unknown parameters on the distribution of a new emergent agent (mutant), as Masanao Aoki(1996) formulated, we can derive emergence of an unknown structure a new structure. In these dynamics, it is important to group by heterogeneous types in a social interaction. We have several subgroup dynamics given by either Dirk Helbing(1995,2003) or Steven Durlauf(1997,2000). On the other hand, when analyzing adaptive behaviors, replicator dynamics directly works well as an intriguing compromise device for many applications. Focusing a side of learning by adaptive agents under the flavor of the Santa Fe institute, economists began to be involved into treating multi-armed bandit problem, in the event, adopting loss minimizations in the face of uncertainty. The rigid worlds of optimization characterized by the duality notions shall be replaced in economics soon.

Leaving from the rigid world of optimization, we will perform some appropriate models of path-independent process as well as path-dependent process on adaptive economic processes in social interactions of heterogeneous agents. By the use of them, microscopic orders of the society can be discussed properly under a certain macroscopic order. A prominent of model on path-independent process will appear from Aoki(1996, 2001), and a prominent one on path-dependent from Arthur(1994).

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Contents

1 A new formulation of microeconomics with specification of macroeconomic field effects

Hoover(2001, 74–75) states:

> The physicist who has successfully reduced the ideal gas laws to the kinetic theory of gases does not then abandon the language of pressure, temperature, and volume then working with gases or try to use momentum, mass, and velocity as the principal phenomenological categories for discussing the macroscopic behavior of gases.
>
> But economists have taken a different track.

Similarly, Weidlich(2000, 11) states concisely:

> In the case of *physical systems* the introduction of the field concept proved to be most important for the deeper understanding of the systemic character. Take the characteristic example of electromagnetic interaction: Particles, the elements of the physical system, posses—besides other properties—both mass and electric charge, and generate in their environment a gravitational as well as an electromagnetic filed. The field contributions of many particles are *superposed* and form a *collective field*.

Hoover(2001, 74) definitely concluds:

> I cannot completely reduce macroeconomics to microeconomics. Microeconomics of the real world necessarily uses macroeconomic models and concepts as an input. The macroeconomy supervenes on the microeconomy but reducible to it.

This suggests that a macroeconomic situation may correspond to multiple microeconomic configurations, namely, a multiplicity of microeconomic situations with a macroeconomic level. The relationship between macroeconomic states and microeconomic configuration has systematically been formulated by Masanao Aoki(1996) and (2002) both of which are published by Cambridge U. P. We need a concept of equilibrium distribution from statistical mechanics.

We first of all start from a linear dynamical system which can be remarkably fitted to a historical data in a sense. This model is quite neutral without specifying any individualistic behavior in view of optimization or non-optimization. But the limitation of this model rather clarifies about what is important in analyzing human behavior

We know that some of differential equations system in either linear or nonlinear can be extremely fit table to the fact under consideration. We can mention Lotka-Volterra dynamics of biology, and Repicator Dynamics who are at present often utilized due to its nonlinearity. The **Lanchester combat model** of an ordinary differential equations system, among them, was widely noticed because of the extremely fittedness to an actual evolution although the system was simply of linear

2 The battle of the IWO-JIMA, 1945

The Iwo-jima is even now a very small island, a part of Japan in the Pacific Ocean. **Iwo** means **sulfur** in Japanese. Iwo-Jima is just an island of volcano which contains much sulfur, namely, an island of sulfur. The offensive and defensive battle of the IWO-JIMA Island between the USA and the Japan forces in the Second World War is well-known as one of the hardest battles in the 20th Century. This had commenced on 19th February 1945 and virtually terminated on 26th March 1945 by the fact of the finally official all-out attack of the army Brigade under the conduct of General Tadamichi Kuribayashi. On the day the USA armed forces declared the termination of the battle. The battle had continued for 36 days.

2.1 The IWO-JIMA battle

The total number of armed forces directly committed by the USA amounted to 73,000. The total shells from warships and artilleries were over 290,000 times. On the other hands, the Japan Brigade was organized by the army of 13,000 soldiers and the navy of 7,450 marines. The headquarters however never decided to reinforce any forces into the IWO-JIMA defense force after the USA's launch attack, only except for the total 75 transport planes from the mainland.

We can trace out the battle development on the daily base by the record noted by Captain Clifford Morehouse, the US Marine. We cannot have any official record on the side of the Japan forces because of annihilation.

	Killed	Wounded	Combat Fatigue	Prisoners	Total[a]
USA	6,821	19,217	2,648		28,686
JAPAN	20,000			1,083	20,000

Table 1: Casualties in the Iwo-Jima, Feb.1945

[a] See Braun(1982,407).

2.2 The formulation by F.W. Lanchester combat model

The IWO-JIMA battle in terms of the Lanchester model has already been became quite famous because the model provided us with a remarkably fittedness to the historical development on the battle. See Braun(1982, Sec.4.5.2).

Since the IWO-JIMA battle can be regarded as the combat between the conventional force vs. the conventional force, we apply the Lanchester combat model of linear ordinary differential equations to this battle. We note this system is of *inhomogeneous* since the system contains the items of reinforcement, namely the armed forces from outside. In either case of homogeneous or inhomogeneous, we can analyze definitely due to *linearity*. We also notice that we could not establish the result without numerical simulation in the case of the guerilla combat due to its *nonlinear* property.

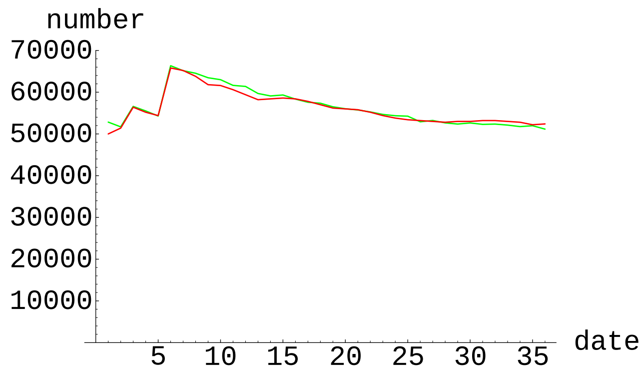

Figure 1: The real shift of forces and the simulation result by the Lanchester model

According to F. W. Lanchester, we can define:

$$\begin{aligned} & \text{the rate of the troop strength} \\ = & \text{ the reinforcement rate} \\ - & \text{ (the operational loss rate + the combat loss rate)} \end{aligned}$$

Let $x_1(t)$, $x_2(t)$ to be the number of American Forces, and Japanese Forces, respectively, starting from the beginning of the attack by the USA.

The loss rate by combat x_1 is presumed to be such all the regular army that x_2 could target within its range shot.

We can express $ax_1(t)$ as $x_1(t)$'s damage by combat, by use of the constant a which is a positive one and also means the combat effectiveness coefficient of x_2. Similarly, we can define b as the loss coefficient of $x_2(t)$, or the combat efficiency by x_1.

The reinforcement rate The *change* of the forces occurs also by reinforcement or withdrawal from readiness.

$$\begin{aligned} f(t) &: \text{ the reinforcement rate to be added to } x_1 \text{ on the } t\text{'th day.} \\ g(t) &: \text{ the reinforcement rate to be added to } x_2 \text{ on the } t\text{'th day.} \end{aligned}$$

The model between the conventional force vs. the conventional force [1]

$$\begin{aligned} \tfrac{dx_1}{dt} &= -ax_2 + f(t) \\ \tfrac{dx_2}{dt} &= -bx_2 + g(t) \end{aligned}$$

[1]Let x_1 to be a state of the guerrilla band, while x_2 to be the conventional force. In this

It was the historical fact that the headquarter of Japanese Forces never decided to reinforce his forces during the battle. Thus the Lanchester Model of the IWO-JIMA decisive battle could be of the following differential equations system:

$$\frac{dx_1}{dt} = -ax_2 + f(t)$$
$$\frac{dx_2}{dt} = -bx_1$$

$$A = \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix};\ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix};\ F = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$$

We then obtain the **initial problem of inhomogeneous linear ordinary differential equations**

$$\frac{dx(t)}{dt} = Ax(t) + F(t),\ x(t_0) = x^0, \tag{1}$$

whose general solution is as follows:

$$x(t) = X(t)X^{-1}(t_0)x^0 + X(t)\int X^{-1}(s)f(s)ds.$$

The explicit solutions of the IWO-JIMA

$$x_1 = \int \cosh\sqrt{ab}\ (s-t)\ f(s)\ ds - \sqrt{\frac{a}{b}}y_0 \sinh\sqrt{ab}\ t \tag{2}$$

$$x_2 = y_0 \cosh\sqrt{ab}\ t + \sqrt{\frac{a}{b}}\int f(s)\sinh\sqrt{ab}\ (s-t)\ ds \tag{3}$$

2.3 A new simulation result if Japanese Forces should decide to reinforce during the battle

As a historical fact, the USA forces had been reinforced by the strength of 19,000 until the 6th days after opening the launch. We can now easily suggest a new result on this battle that the Japan forces could remain the same strength before opening the battle, even after the 50 days battle, if the headquarter of Japan should have realized to reinforce by the same number of strength as the USA's increase, until the 7th day after the opening battle, and on the contrary, the USA forces should be annihilated.

case,

$$\frac{dx_1}{dt} = -cx_1x_2 + f(t)$$
$$\frac{dx_2}{dt} = -dx_2 + g(t)$$

If x_1 were a guerrilla band, x_1 could not come all into the x_2's sight and be not all targeted. The offence by x_2 on a particular field could not have a certain achievement. Thus the galliard band x_1 has the rate of damage by combat as a nonlinear product $cx_1(t)x_2(t)$. It is noted that the dynamical system of Lanchester's combat model may be of *nonlinear* form. c is a positive constant. As the number of the guerrilla is fewer, the lower the damage rate could be. But the guerrilla will rapidly lose its advantage by increasing its number of guerrilla member as well as also by the enemy's increase of forces.

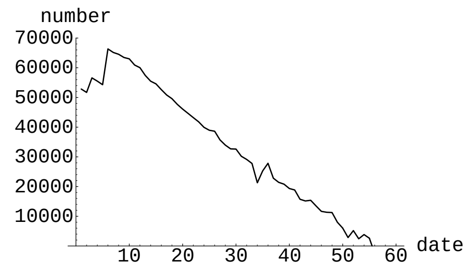

Figure 2: The activity development of the USA offence forces

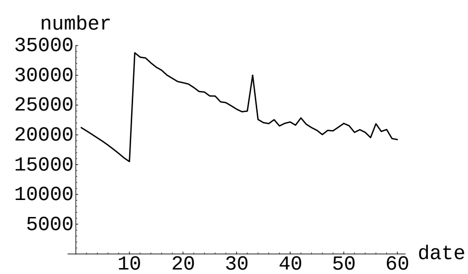

Figure 3: The activity development of the Iwo-jima defense forces

2.4 Difficulties to estimate the combat effectiveness in view of inside-observers

Our new simulation result demonstrates about what a decisive factor in the Lanchester combat model is. It is noteworthy that the most important and difficult thing is to estimate the coefficient a, namely the damage rate of x_1 or the combat efficiency of x_2, as well as the coefficient b of the counterpart. Without these right specifications, we cannot describe the real development of the battle or the activities of both combating forces.

These however just are given *ex post* by the use of the historical data. If we were able to employ these, we could always optimize ours strategy. We however failed to optimize our strategy because we were not able to know exactly these coefficients in front of uncertainty.

It may be true that the battle dynamics seems to be a simple dynamics on the macroscopic surface, if we are permitted not to specify the microscopic configurations. Thus the Lanchester model can be successful to trace out extremely precisely the real process of the battle dynamics. But this description only is given *ex post*.

We can also apply this kind of incapability to our ordinary prediction to know about the real efficiencies of our ordinary life of business and other activities. Even if we knew exactly a sufficiently enough set of physical information on the enemy, we could not estimate rightly her capabilities.

3 The 2-armed bandit problem

One-armed bandit is a jargon on a slot machine. The 2-armed bandit problem means the choice problem faced to two objectively known but subjectively unknown slot-machines.[2]

3.1 Choice on two slot machines with overlapping yield distributions

We illustrate this problem by the use of Holland(1992). In this setting a player must be faced to the choice problem of two slot machines with overlapping yield distributionsis. We cite this situation from Holland(1992,76–77), though at lengty.

> If it could be determined through some small number of trials which of ξ and ξ' has the higher mean, then from that point on all trials could be allocated to that random variable. Unfortunately, unless the distributions are non-overlapping, no finite number of observations will establish *with certainty* which random variable has the higher mean.

[2] See Bellman(1961).

Given $\mu_\xi > \mu_{\xi'}$ along with a probability $p > 0$ that a trial of ξ' will yield an outcome $x > \mu_{\xi'}$, there is still a probability p^N after N trials that *all* of the trials have had outcomes exceeding μ_ξ. A fortiori their average $\hat{\mu_{\xi'}}$ will exceed μ_ξ with probability at least p^N, even though $\mu_{\xi'} < \mu_\xi$.

Here **the tradeoff between gathering information and exploiting** it appears in its simplest terms. To see it in exact form let $\xi_{(1)}(N)$ name the random variable with the highest observed random payoff rate(average per trial) after N trials and $\xi_{(2)}(N)$ to be the other random variable. For any number of trials n, $0 \leq n \leq N$, allocated to $\xi_{(2)}(N)$ (and assuming overlapping distributions) there is a positive probability, $q(N-n,n)$, that $\xi_{(2)}(N)$ is actually the random variable with the highest mean, max $(\mu_\xi, \mu_{\xi'})$.

The two possible sources of loss are:
(1)The *observed* best $\xi_{(1)}(N)$ is really second best, whence the $N-n$ trials given $\xi_{(1)}$ incur an (expected) cumulative loss

$$(N-n)|\mu_\xi - \mu_{\xi'}|.$$

(2)The *observed* best is in fact the best, whence the n trials given $\xi_{(2)}$ incur a loss;

$$n|\mu_\xi - \mu_{\xi'}|.$$

This occurs with probability

$$(1 - q(N-n,n)).$$

The total expected loss for any allocation of n trials to $\xi_{(2)}$ and $N-n$ trials to $\xi_{(1)}$ is thus

$$L(N-n,n) = [q(N-n,n)(N-n) + (1-q(N-n,n)n]|\mu_\xi - \mu_{\xi'}|.$$

We shall soon see that, for n not too large, the first source of loss decreases as n increases because both $N-n$ and $q(N-n,n)$ decrease. At the same time the second source of loss increases. By making **a tradeoff between the first and the second sources of loss**, then, it is possible to find for each N a value $n^*(N)$ for which the losses are minimized; i.e.,

$$L(N-n^*,n^*) \leq L(N-n,n) \quad \text{for all } n \leq N.$$

3.2 Implications of the 2-armed bandit problem

The theorem on the 2-armed bandit choice suggests that:

> Detecting individualistically the best choice by *learning* under *uncertainty*, a microeconomic agent always must be afraid of incurring loss.

By this fact, a macroeconomic state will be at least *comprised of* the two different groups:

3-1 The successful group for the best choice

3-1 The failure group to grasp the best

The latter must *imitate* or *learn* the former group. This may create a social stochastic dynamics. In other words, an *adaptive* process in social interaction can be well defined by the following stochastic model.

4 A stochastic economic system by Masanao Aoki

We define a Markov chain X_t on the state space S which dynamically describes a flux of probabilities. In a state j at time t, we can imagine a probability $P_j(t) = \Pr(X_t = j)$. We suppose that there are a number of independent agents, each of whom is in one of finite microeconomic states, and its each state evolves according to the master equation

$$\frac{dP_j(t)}{dt} = \sum_{k \neq j} [P_k(t) w_{kj} - w_{jk} P_j(t)] \; j \in S. \tag{4}$$

where w_{kj} denotes the transition rates from state j to state k, or the in-flow rates, and w_{jk} the rates from k to j, the out-flow rates.

4.1 A Gibbs distribution in terms of the master equation

The system of two states: Then the economy is composed by

$$S = \{a, b\}.$$

Given the nonnegative transition rates $w_{ab}, w_{ba} \geq 0$, the master equation system is constituted by the next two ordinary differential equations:

$$\frac{dP_a(t)}{dt} = P_b(t) w_{ba} - P_a w_{ab}, \tag{5}$$

$$\frac{dP_b(t)}{dt} = P_a(t) w_{ab} - P_b w_{ba}. \tag{6}$$

Here

$$P_a(t) + P_b(t) = 1.$$

In this case, given an initial condition $P_a(0)$, and $w_{ab} + w_{ba} > 0$, it follows:

$$\pi_a = \lim_{t \to \infty} P_a(t) = \frac{w_{ba}}{w_{ba} + w_{ab}}. \tag{7}$$

In general, the master equation may be

$$\frac{\partial P(s,t)}{\partial t} = \sum_{s \neq s'} P(s,t)w(s' \mid s,t) - \sum_{s \neq s'} P(s',t)w(s \mid s',t).$$

In the stationary state or in equilibrium, the probability in-flows and out-flows balance at every state:

$$\pi_j \sum_{k \in S} w_{jk} = \sum_{k \in S} \pi_k w_{kj} \ \forall j \in S, \tag{8}$$

$$\pi_j \geq 0, \sum_{j \in S} \pi_j = 1. \tag{9}$$

If there exist a j, and a π_j, we call the distribution

$$\{\pi_j, j \in S\}$$

"**an equilibrium distribution**."

If the probability flows balance for every pair of states, it then holds **the detailed balance condition**:

$$\pi_j w_{jk} = \pi_k w_{kj}.$$

Given an **irreducible** Markov chain, for any state s_j, we can find a finite sequence

$$\{s_1, s_2, \cdots, s_j\}$$

starting from some initial state s_0. If the detailed balance condition holds, it follows *a Gibbs distribution*

$$\pi_j = \pi(s_0) \prod_{i=0}^{j-1} \frac{w_{i+1,i}}{w_{i,i+1}}.$$

Here

$$\pi(s) = K \exp[-U(s)]$$

with

$$U(s_j) - U(s_0) = \sum \ln \frac{w_{i+1,i}}{w_{i,i+1}}$$

implies $U(s)$ to be *a potential.*

The probability distribution is *independent of paths* from s_0 to s_j due to the property of Markov chain process. See Aoki(1996, 118).

4.2 Multiplicity of microeconomic configurations

The total number of states is set N. The $n(t)$ indicates the state variable. We have ${}_N\mathrm{C}_n$ ways to realize **the same** $\frac{n}{N}$. In other words, the equilibrium distribution gives the idea of multiplicity of micro-configuration that produces the same macro value. By use of $\frac{n}{N}$, we employ a variable x such that

$$\frac{x+1}{2} = \frac{n}{N}. \tag{10}$$

x measures the fraction from the median. It follows

$$\mathrm{d}x = \frac{2}{N}. \tag{11}$$

The base e Shannon entropy, by use of the above variable x, is given by

$$H(x) = -\frac{1+x}{2}\ln\frac{1+x}{2} - \frac{1-x}{2}\ln\frac{1-x}{2}. \tag{12}$$

It also is verified

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{1}{2}\ln\frac{1-x}{1+x}. \tag{13}$$

The approximation formula thus gives

$${}_N\mathrm{C}_n = \exp\left[NH\left(\frac{x+1}{2}\right)\right] + O(\frac{1}{N}). \tag{14}$$

4.3 The microeconomic birth- and -death process in a discrete form

Now suppose that the number of microeconomic agents in this subclass is reduced by one(a death), and the number of agents increases by one (a birth). It then follows the master equation in the discrete case:

$$P_{t+1}(n) = W_{n-1,n}P_t(n-1) + W_{n+1,n}P_t(n+1) + W_{n,n}P_t(n), \tag{15}$$

where $W_{n,n-1}$ is the probability of transition rate from state n to $n-1$, and where $W_{n,n+1}$ denotes the probability of transition from n to $n-1$.

Let the time step to be small enough.

$$W_{n,n} = 1 - W_{n,n+1} + W_{n,n-1}. \tag{16}$$

expresses the probability that the number of agents may remain the same, normally as assumed to be positive.

The detailed balance condition

$$\Pi(n)W_{n,n+1} = \Pi(n+1)W_{n+1,n} \tag{17}$$

gives the equilibrium probability Π's. The equilibrium probability distribution will form a *Gibbs distribution.*

The solution of the above difference equation gives

$$\Pi(n) = \Pi(0) \prod_{k=1}^{n} \frac{W_{k-1,k}}{W_{k,k-1}}. \tag{18}$$

In the simplest case, a death shows $W_{n-1,n}$, while a birth $W_{n,n+1}$. Thus, for some constants μ, λ, it holds

$$\begin{aligned} W_{n,n-1} &= \mu n \\ W_{n,n+1} &= \lambda(N-n). \end{aligned}$$

In the following, we set for convenience

$$\mu = \lambda.$$

Let $u_i(x)$ to be a *perceived random benefit*

> over some unspecified planning horizon of adopting alternative i when fraction x of agents are using it. (Aoki 1996,138).

Given the two choices 1 and 2, we define the **benefit difference** between two states:[3]

$$\triangle G = u_1(x) - u_2(x). \tag{19}$$

$$\eta_1(x) = \Pr(u_1(x) \geq u_2(x)). \tag{20}$$

Taking some **nonlinear effects** into account, we assume

$$W_{n,n-1} = N\left(1 - \frac{n}{N}\right)\eta_1 \frac{n}{N}, \tag{21}$$

$$W_{n,n+1} = N\left(\frac{n}{N}\right)\eta_2 \frac{n}{N}. \tag{22}$$

The **potential** is given

$$\begin{aligned} & -\beta N U\left(\frac{n}{N}\right) - \ln Z \\ = {} & \ln \Pi(0) + \ln_N \mathrm{C}_n + \sum^{n} \ln \frac{\eta_1(k/N)}{\eta_2(k/N)}. \end{aligned} \tag{23}$$

[3]In a continuous case, we may make

$$\eta_1(x) = \frac{\exp[\beta \triangle u(x)]}{\exp[\beta \triangle u(x)] + \exp[-\beta \triangle u(x)]}.$$

Here β expresses a degree of uncertainty over the concerned system.

Here

$$Z = \sum_n \exp[-\beta NU\left(\frac{n}{N}\right)] < \infty \tag{24}$$

is the **partition function**.

We set

$$\ln\frac{\eta_1(k/N)}{\eta_2(k/N)} = 2\beta\triangle G\left(\frac{k}{N}\right).$$

According to Aoki(1996, 141), we can interpret β with

> **the degree of uncertainty** in the economic environment surrounding agents, or the general level of economic activity.

Then **the equilibrium probability** can be expressed by

$$\Pi(n) = \frac{\exp[-\beta NU\left(\frac{n}{N}\right)]}{Z} \tag{25}$$

Neglecting the items $O(1/N)$, it follows:

$$U\left(\frac{n}{N}\right) = -\frac{2}{N}\sum_{k=1}^{n}\triangle G\left(\frac{k}{N}\right) - \frac{1}{\beta}H\left(\frac{n}{N}\right). \tag{26}$$

Hence,[4]

$$U(x) \approx -\int \triangle G(y)\mathrm{d}y - \frac{1}{\beta}H(x) \tag{27}$$

$$= -G(x) - \frac{1}{\beta}H(x). \tag{28}$$

Taking the relations obtained in the above preliminaries into consideration,

$$\frac{\mathrm{d}U(x)}{\mathrm{d}x} \approx -\triangle G(x) - \frac{1}{2\beta}\ln\frac{1-x}{1+x}. \tag{29}$$

In the equilibrium in which the potential never be changed, it holds

$$2\beta\triangle G(x) = \ln\frac{1+x}{1-x}. \tag{30}$$

This is just a *Gibbs distribution* derived from the discrete time formulation. This approach rather demonstrates a *qualitative* prediction in social interaction of heterogeneous agents by use of the concept of **equilibrium distribution** or an ensemble mean.

[4] If $\triangle G = 0$, it holds the simple birth-and -death process.

4.4 Choice under uncertainty and the multiple local equilibria: subgroup dynamics

When all the members in this system succeed to maximize their utilities, the system can reach the level of $u(a)$. If some of the members failed to maximize utility, the level of the system could go down to $u(b)$. In other words, the system may be constituted by the two groups, one of which is successful for optimization, the other of which failed optimization. We may construct the average utility by

$$pu_1 + (1-p)u_2,$$

where p is the ratio of the failure group.

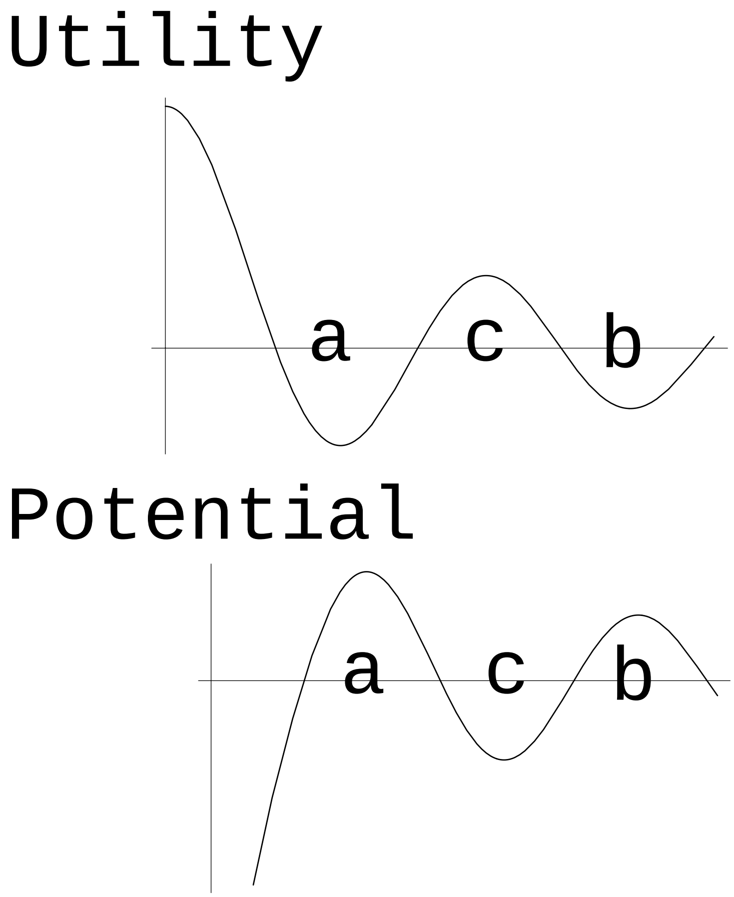

Figure 4: Utility and Potential

The state point a and b are the equilibrium points. Between them we imagine a Markov process by employing the transition rates[5]

$$\begin{aligned} w_{ab} &= \mathrm{e}^{-\beta(V+v)} \\ w_{ba} &= \mathrm{e}^{-\beta V}, \ \beta > 0. \end{aligned}$$

On the side of potential, we assume

$$v = V_b - V_a > 0.$$

It then follows from the equation (7) that

$$\pi_a = \frac{w_{ba}}{w_{ba} + w_{ab}} = \frac{1}{\mathrm{e}^{\eta v + 1}} \tag{31}$$

$$\pi_b = 1 - \pi_a = \frac{\mathrm{e}^{\beta v}}{\mathrm{e}^{\beta v + 1} + 1}. \tag{32}$$

If v is taken small enough,

$$\pi_a \approx \pi_b \to \frac{1}{2}.$$

This system then has equally **two local equilibria**. Either of them can equally occur.

First passage times of unstable dynamics Measured in the levels of potential, the figure suggests an unstable situation of the macroeconomic states. There are three critical points arranged as

$$\phi_a \geq \phi_b \geq \phi_c.$$

Here ϕ is a macroeconomic state. We take ϕ_a the initial state and ϕ_c the final state. The final state may be interpreted an absorbing state.

van Kampen(1992) and Aoki(1996, Sec. 5.11) showed that a **passage time** τ_{ca} from c to a is given by

$$\tau_{ca} \propto \mathrm{e}^{\beta N V(v)}.$$

> **the critical values** of the potential are the critical points of the equilibrium probability distribution $[\Pi(n)]$ and are the same as **the critical points** of the macroeconomic dynamics. (Aoki 1996, 142)

5 A generalization of path dependent process in the context of W. B. Arthur

The old theory of utility in economics thus now is to be replaced with a new one of statistical mechanics, as we discussed according to Aoki(1996) and Aoki(2001). In the above formulation, the probability distribution is *independent of paths* from s_0 to s_j due to the property of Markov chain process.

[5] As β changes, the number of equilibrium changes.

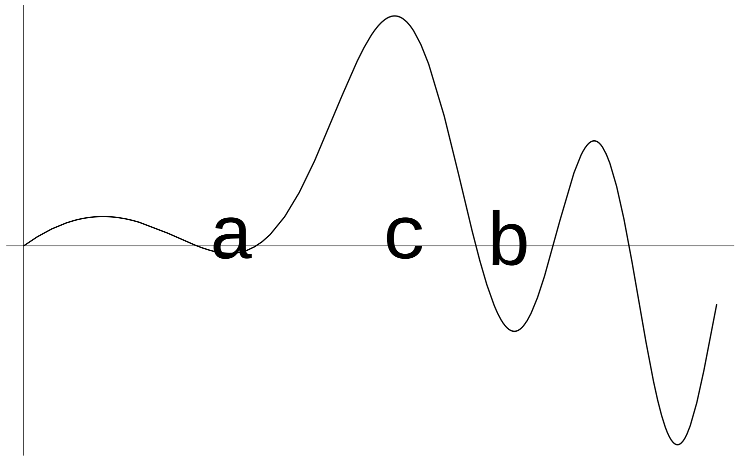

Figure 5: Critical states in terms of potential

5.1 A study on path deendent process

When we study social interactions, we shall have the next two kinds of issue:

5-1 A justification comes from that it is evident that there are many subgroups in our society, namely, we cannot regard the society as an aggregate of homogeneous agents..

5-2 Another one is that a random selection of firms as supported by an active deregulation policy can of necessity cause a or a few firms to the *winner-take-most*.

The former **5-1** is closely connected with the domains of Aoki(1996, 2001). While the latter **5-2** indicates a reinforcement process or positive feedback process as Arthur(1994) were concerned with.

> The result of deregulation over the long term has been a steady decline in large carriers, from 15 in 1981 to around 6 at present[in 1996]. Some routes have become virtual monopolies, with resulting higher fares. None of this was intended. But it should have been predicted-given increasing returns. (Arthur 1996, Section: What about Service Industries?)

Arthur(1988)[in Arthur(1994,129)] also says:

> Economics agents may be influenced by neighbor agents' choices. Puffert(1988) examines historical competitions between railroad gauges where rail companies found it advantageous to adopt a gauge that neighboring railroads were using. Spatial mechanisms have parallels

> with Ising models and renormalization theory in physics(Holly 1974) and with voter models in probability theory(Liggett 1979; Föllmer 1979). In economics very little work has yet been done on spatial self-reinforcing mechanisms.[6]

Arthur, when arguing the property of **reinforcement mechanism** in general, not in spatial interests, rather suggests the use of **Polya Urn Process** of Arthur, Ermoliev, and Kaniovski(1983)[in Arthur(1994, Chapter 10)].

In a generalized version of Polya urn process to allow for a more number of agent(color) than two, a random **vector** X whose components are proportions of each agent, instead of a numerical value(ratio), is to be introduced.

Polya urn process Polya and Eggenbarger(1923) originally showed the following process.

> Think of an urn of infinite capacity to which are added balls of two possible colors — red and white, say. Starting with one red and one white ball in the urn, add a ball each time, indefinitely, according to the rule: Choose a ball in the urn at random and replace it: if it is red, add a red; if it is white, add a white. Obviously this process has increments that are path-dependent — at any time the probability that the next ball added is red exactly equals the proportion. We might then ask: does the proportion of red (or white) balls wander indefinitely between zero and one, or does a strong law operate, so that the proportion settles down to a limit, causing a structure to emerge? Arthur(1994)

5.2 Generalized version of Polya urn process

In a **Generalized Version of Polya Urn Process** to allow for a more number of agent(color) than two, a random **vector** X whose components are proportions of each agent, instead of a numerical value(ratio), is to be introduced.

> **Theorem 3.1 of Arthur(1994, Chapter 10, 189–190).** Given continuous urn functions $\{q_n\}$, suppose there exists a Borel function $q : S \to S$, constants a_n and a Lyapunov function$\nu : S \to R$ such that:
> (a)$sup_{x \in S} \|q_n(x) - q(x)\| \le a_n, \sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$
> (b)The set$B = \{x : q(x) = x, x \in S\}$ contains a finite number of connected components.
> (c)(i)ν is twice differentiable.
> (ii) $\nu(x) \ge 0, x \in S$
> (iii) $\langle q(x) - x, \nu_x(x) \rangle < 0, x \in S \backslash U(B)$ where $U(B)$ is an open neighborhood of B and $<, >$ represents inner products.

[6]See Anderson, Arrow, Pines(1988, 9–31).

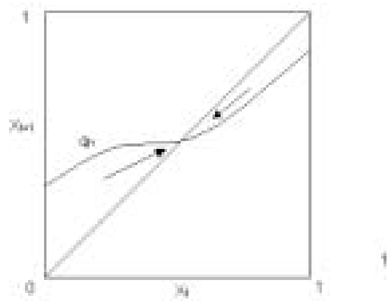

Figure 6: Nonlinear Polya function

Then x_n converges to a point of B or to the border of a connected component.

5.3 Avatamsaka game in view of a nonlinear Polya urn process

Avatamsaka Situation We imagine "driving a car" in a busy place. When we wish to enter into a main street from a side road or a parking area along the road, it must be much better for us that someone can give way for us although someone cannot gain any more by giving us whom someone might never meet. This kindness done by someone implies our gain. A symmetrical situation in replacing someone with us holds the same relation. Anybody who gives a kind arrangement to someone, can never be guaranteed to gain from another. If nobody gave way for someone, the whole welfare on the roads should be extremely worsen. We call such a situation *Avatamsaka Situation.* [7]

Strategies	Player B	
(i)Coordination game		
Player A	Strategy 1	Strategy 2
Strategy 1	(a, a)	(b, c)
Strategy 2	(c, b)	(d, d)
(ii)Avatamsaka game		
Player A	Defection	Cooperation
Defection	(0, 0)	(1, 0)
Cooperation	(0, 1)	(1, 1)

Table 2: Avatamsaka game

[7]See Aruka(2001a) for the details of this game.

5.4 International results of Avatamsaka experiment

We reproduce our international results of Avatamsaka experiment. The experiments all were done by the use of network program at the author's computer server in Chuo University, Tokyo, Japan. Those games except for the case of Tokyo May 2, 2001 have been done usually for 10 rounds each in total. The first subgroup of one experiment on the web was held at the PC- Pool, J.-W. Goethe Universitaet (Frankfurt University), Frankfurt am Main for Dec.12, 2000.

The second subgroup of two experiments has been collected from Japanese subjects in the Multimedia Room 2109, Chuo University, Tokyo. The subjects in the experiment of January 24, 2001 were sampled from graduate students of commerce and economics. On the contrary, the subjects of the other experiment of May 2, 2001 consisted of the age distribution of almost 19 years old students in the Faculty of Commerce, Chuo University.

The third subgroup of two experiments was conducted by Professor Stephen Guastello, and Mr. Robert Bond Jr., professor's assistant, the Department of Psychology, February 16 and 23, 2000, Marquette University, Milwaukee, USA. We have achieved a lot of experiments other than these but a few complete collections of experimental data. The failures of experiments on the web led us to much improvements of our program. It is also noted that the age distribution of his students is quite similar to our sample students of May 2, 2000 in Tokyo.

The results can be summarized altogether in Table 3 as follows:[8]

Average*	Frankfurt	Tokyo I	Tokyo II(1)[a]	Tokyo II(2)[b]	Milwaukee I	Milwaukee II
Pay-off	0.731	0.709	0.527	0.502	0.509	0.557
C-ratio	0.738	0.713	0.53	0.503	0.509	0.559

Table 3: International results of Avatamsaka experiment

* Pay-off: total gains to total rounds; C-ratio: the times of cooperation taken to the total rounds.

[a] Tokyo II(1) shows the average value on the first 10 stages.

[a] Tokyo II(2) shows the average on the entire 20 stages.

5.5 Application to Avatamsaka game

[9] Arthur called a limit **proportion** in such a path dependent stochastic process an asymptotic **structure**. As long as the classical proposition is considered,

[8] Aruka(2002) dealt with the statistical tests on these results in details and attained the conclusion: "[i]n summary, it seems us capable to eliminate the effects of country factor on the strategically collective deployment for each game, if we could properly adjust disturbances due to the initial conditions."(Aruka 2002,105)

[9] This subsection is largely cited form Aruka(2001c, 158–161).

we shall be faced to a limit of proportion, i.e., a simple structure, as proved in Polya(1931). In the classical Polya example, we have a restriction such that the probability of adding a ball of type j exactly equals the proportion of type j. In the following we impose the next assumptions for a while.

5-A1 We suppose there are a *finite* number of players, $2N$, for instance, who join by each pair into our Avatamsaka Game.

5-A2 We *identify* the ratio of cooperation, or C-ratio for each player *with* the proportion of the total possible gains for each player.

A generalization We just notice that we regard a red ball as a gain of Player A who loves employing Defection Strategy in our two person game, while a white ball as a gain of Player B who loves employing Cooperation Strategy, for instance. Player A can then continue to increase his gain if the proportion of his gain is kept higher than Player B. It could be possible, provided that Player B still clung to his original policy. The Polya original restriction never allow for Player B to change his mind. A *nonlinear* inspection for Players may be taken for account to obtain a more realistic result. Cooperation *by reinforcement* grows herself, if there is a greater ratio of cooperation. Defection can however grow by exploiting a greater population of cooperation. We have usually **10 players** in our game experiment. It may be desirable that the Polya urn may contain **ten colors** in order to analyze **a limit proportion** of the times of Defection to Cooperation of this game.

A generalized Polya urn process It is quite interesting to learn that Arthur, a pioneer of **Economics of Complexity** has allocated *one third* of the total pages in his reputable book titled *Increasing Returns and Path Dependence in the Economy*, Arthur(1994) for generalization of Polya process. That famous term like "lock-in" could not be broadly accepted by specialists if a generalization of Ploya process should not be successful. In a generalized version of Polya urn process to allow for a more number of agent(color) than two, a random **vector** X whose components are proportions of each agent, instead of a numerical value(ratio), is to be introduced.

A generalization of Path dependent process In the line of Arthur, Ermoliev and Kaniovski, we describe **a generalization of path dependent process** in terms of our *repeated game*. Let X_i to be a proportion in the total possible points which Player i gains. The initial total *potential* of gains for each player is defined $2N-1$. In the next period 2, the total maximal gains will grow by the number of players $2N$. In the period n, therefore, the proportion for Player i at time n will be

$$X_i^n = \frac{b_i^n}{(2N-1)n} \ . \tag{33}$$

Suppose such *a sample space* that

$$X_i : \Omega \rightarrow [0,1] \quad (34)$$

$$\omega \in \Omega \rightarrow X_i(\omega) \in [0,1] \quad (35)$$

We can then have a probability $x = X_i(\omega)$ for a sample ω. If a sample implied an experimental result, the sample space Ω could make out *a psychological space* for each agent to give a next move. The probability x may depend on a random variable X_i. If we have $2N$ agents, it then follows a random vector of proportions X^n of period n:

$$X^n = (X_1^n, X_2^n, ..., X_{2N}^n) \ . \quad (36)$$

where it is assumed that each element X_i^n mutually is *independent*, in other words, each agent is *psychologically independent*. Players start at a vector of the initial gains distribution

$$b^1 = (b_1^1, b_2^1, ..., b_{2N}^1) \ . \quad (37)$$

Let $q_i^n(x)$ define a probability of player i to *earn* a point *by means of Strategy C* in period n. Thus a series of proportions

$$\{X^1, X^2, X^3, ..., X^n\} \ . \quad (38)$$

will be generated after the iteration of n times.

In the following, q^n behaves as a rule for *a mapping* from X^n to X^{n+1}. Consider a one-dimensional dynamics.

$$q_i(X_i)(\omega) \in [0,1] \quad (39)$$

$$q_i(x) : \Omega \rightarrow [0,1] \quad (40)$$

If X^n, as a random variable giving a proportion x, appeared at a low level, a value of X^{n+1} might be greater than the previous value of X^n. This case is implied by **5-A2** that a low level of proportion in gain can induce an increase of the ratio of cooperation in an individual strategy. If q is *nonlinear*, a *nonlinear* dynamics may be generated.

A nonlinear Polya process We have already defined a function q for a $x = X_i(\omega)$, on one hand. Now, on the other hand, we define a function β for the same ω:

$$\beta_i(X_i)(\omega) \in \{0,1\} \quad (41)$$

$$\beta_i(x) : \Omega \rightarrow \{0,1\} \quad (42)$$

It then follows, if $x = X_i^n(\omega)$,

$$\beta_i^n(x) = 1 \quad \text{with probability given by} \quad q_i^n(x) \quad (43)$$

$$\beta_i^n(x) = 0 \quad \text{with probability given by} \quad 1 - q_i^n(x) \quad (44)$$

Each player might have a maximum gain of $2N-1$ from the $2N-1$ matches. Hence **the dynamics** for addition of point by Player i may be written:

$$b_i^{n+1} = b_i^n + (2N-1)\beta_i^n(x) \ . \tag{45}$$

b_i^n then shows an *accumulated* gain at time n through the iteration of game for Player i. Thus **the nonlinear evolution** of the proportion for Player i will, by substitution of X^n, be given in the form of

$$X_i^{n+1} = X_i^n + \frac{1}{n+1}(\beta_i^n(x) - X_i^n) \ . \tag{46}$$

Reformulate this equation in view of

$$\xi_i^n(x) = \frac{1}{n+1}(\beta_i^n(x) - q_i^n(x)) \ . \tag{47}$$

We then obtain

$$X_i^{n+1} = X_i^n + \frac{1}{n+1}(q_i^n(x) - X_i^n) + \xi_i^n(x) \ . \tag{48}$$

Notice that **the conditional expectation** of ξ_i^n is zero. Thus we have reached **the expected motion**:

$$E(X_i^{n+1} \mid X^n) = X_i^n + \frac{1}{n+1}(q_i^n(x) - X_i^n) \ . \tag{49}$$

from which we can easily verify that motion tends to be directed by the term $q_i^n(x) - X_i^n$. Arthur and others call this process *a nonlinear Polya process* when the form of q is *nonlinear*.

The above process is just *a martingale*. By a direct application of the theorems of Arthur, Ermoliev and Kaniovski, we could justify a convergence of the C-ratios in the community *not* to the extreme points like $0, 1$.

> *Players shall initially be motivated by the behaviors of the other players. Eventually, however, players' behavior could be independent from the others.*

An effect of **path dependency** may sometimes be discussed in a connection with econophysics.

References

Anderson, P., Arrow, K., and Pines, D.(eds.)(1988) The Economy as an Evolving Complex system: Studies in Sciences of Complexity, Addison-Wesley,Redwood CA

Aoki, Masanao(1996) New Approaches to Macroeconomic Modeling: Evolutionary Stochastic Dynamics, Multiple Equilibria, and Externalities as Field Effects, Cambridge U. P., New York

Aoki, Masanao(2002) Modeling Aggregate Behavior and Fluctuations in Economics: Stochastic Views of Interacting Agents, Cambridge U. P., New York.

Arthur, W. B.(1994) Increasing Returns and Path Dependence in the Economy, University of Michigan Press, Ann Arbor.

Arthur, W. B. (1996) Increasing returns and the new world of business, Harvard Business Review 74(July-Aug):100–109

Aruka, Y.(2001a) Avatamsaka game structure and experiment on the web. In Aruka, Y.(ed.) Evolutionary Controversies in Economics: A New Transdisicplinary Approach, Springer, Tokyo(115–132)

Aruka, Y.(2001b) The law of consumer demand in Japan: a macroscopic microeconomic view." In Takayasu, H.(ed.)(2001) Empirical Science of Financial Fluctuations: The Advent of Econophysics, Springer, Tokyo(294–303).

Aruka, Y.(2001c) Some statistical implications of Avatamsaka game experiment. In Namatame, A., et.al(eds.), Agent-based Approaches in Economic and Social Complex Systems, IOS press, Amsterdam(101–111)

Aruka, Y.(2002), Avatamsaka game experiment as a nonlinear Polya urn process. In Terano, T., A. Namatame et.al., Exploring New Frontiers on Artificial Intelligence, Springer, Berlin(153–161)

Bellman, R.(1961) Adaptive Control Processes: A Guided Tour, Princeton U. P., Princeton

Braun, M, Ordinary Differential Equations, 3rd Ed., Springer, New York.

Durlauf, S.N.(1997) Statistical mechanics approaches to socioeconomic behavior. In Arthur, W. B., Durlauf, S. N., and Lane, D.A.(eds.) The Economy as an Evolving Complex System II, Addison-Wesley, Mass.(81–104)

Durlauf, S. N.(2000): A Framework for the Study of Individual Behavior and Social Interaction. SSRI Working Paper Series(Social Systems Research Institute of the Economic Department), the University of Wisconsin, Madison.

Föllmer, H. (1974) Random economies with many interacting agents, Journal of Mathematical Economics 1:51–62
Bellman, R.(1961) Adaptive Control Processes: A Guided Tour, Princeton U. P., Princeton
Helbing, D. (1995) Quantitative Sociodynamics: Stochastic Methods and Models of Social Interaction Processes, Kluwer Academic Press, Dordrecht.

Helbing D (2003) Dynamic decision behavior and optimal guidance through information services: models and experiments. In Schreckenberg M, Selten R(eds.) Human Behaviour and Traffic Networks. Springer, Berlin (in print

Holland, J. H.(1992) Adaptation in Natural and Artificial Systems, MIT Press, Cambridge, Mass.

Hoover, K.D.(2001) The Methodology of Empirical Macroeconomics, Cambridge UP, Cambridge UK.
Polya, G.(1931) Sur quelques Points de la Théorie des Probabilités, Annales de 1'Institut Henri Poancaré 1:117-161
Polya, G., and Eggenberger, F.(1923) Uëber die Statistik verketteter Vorgange, Zeitschrift für Angewandte Mathematik und Mechanik 3:279-289
van Kampen, N. G.(1992) Stochastic Process in Physics and Chemistry, Rev. Ed, North-Holland, Amsterdam
Weidlich, W.(2000) Sociodynamics: A Systematic Approach to Mathematical Modeling in the Social Sciences, Harwood Academic Publisher, London